ON THE TRANSITION FROM SUBSONIC TO SUPERSONIC VELOCITIES IN LAVAL NOZZLES

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This investigation deals with three-dimensional mixed flows of an ideal gas in Laval nozzles. The study is concerned particularly with the form of the surface of transition in the case when the velocity at the center of the flow approaches the velocity of sound, while the derivative of the velocity in the direction of the canal axis at that point vanishes. A theorem is derived which is a generalization for three-dimensional motion of a well-known theorem of Frankl and Görtler, valid in the cases of plane-parallel and axisymmetrical gas streams [1,2]. On the basis of this theorem two possible types of flows in the neighborhood of the throat of a nozzle, and the possibility of transition of one type into the other are discussed.

1. The equations describing three-dimensional irrotational isentropic flows of an ideal gas in a cartesian coordinate system have the form

$$(a^2 - u^2)\frac{\partial^2\Phi}{\partial x^2} + (a^2 - v^2)\frac{\partial^2\Phi}{\partial y^2} + (a^2 - w^2)\frac{\partial^2\Phi}{\partial z^2} - 2uv\frac{\partial^2\Phi}{\partial x\,\partial y} - 2uw\frac{\partial^2\Phi}{\partial x\,\partial z} - 2vw\frac{\partial^2\Phi}{\partial y\,\partial z} = 0 \qquad (1.1)$$

$$2a^2 = (\varkappa + 1) - (\varkappa - 1)\,q^2 \qquad (1.2)$$

$$q^2 = u^2 + v^2 + w^2, \qquad u = \frac{\partial\Phi}{\partial x}, \qquad v = \frac{\partial\Phi}{\partial y}, \qquad w = \frac{\partial\Phi}{\partial z} \qquad (1.3)$$

where a is the velocity of sound, κ is the exponent of the Poisson adiabatic, Φ is the potential, u, v and w are the vector components of the velocity q along the x-, y- and z-axes respectively. The system of units is chosen in such a manner as to make the magnitude of critical velocity $a = 1$.

Using Equations (1.2) and (1.3), we will write Equation (1.1) in the form

$$[(\varkappa+1)-(\varkappa-1)q^2]\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right)=\frac{\partial\Phi}{\partial x}\frac{\partial q^2}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial q^2}{\partial y}+\frac{\partial\Phi}{\partial z}\frac{\partial q^2}{\partial z} \tag{1.4}$$

It is known that in the region of the throat of the Laval nozzle either of two types of mixed flows may exist. In the first, in the subsonic velocity field, there are local supersonic zones, adjacent to the walls of a canal (Taylor's flows); in the second, the velocity varies from subsonic to supersonic as it passes through the throat of the nozzle (Meyer's flow). Plane and axisymmetrical gas flows of both types were the subjects of investigation of several papers [1-10]; analogous three-dimensional problems were investigated by the author [11,12].

In the present paper we are investigating the conditions under which the supersonic zones which are adjacent to the walls of the nozzle are joined on the axis of the channel and how the transition from mixed three-dimensional gas motion of the Taylor type to the Meyer flow takes place. For this we shall investigate a stream which will be assumed to have two mutually perpendicular planes of symmetry in the region of the sonic surface. The straight line along which these planes intersect coincides with the axis of the channel and we shall also let it be coincident with the x-axis. We will assume in what follows that the direction of the velocity vector of the main flow of a gas is along this axis, also at the origin of the coordinate system its magnitude approaches the velocity of sound. Thus, the point $x = y = z = 0$ is the point of intersection of the channel axis with the surface of transition, i.e. it is the center of the flow [6].

In [8,9,12] solutions were obtained describing a limiting case of the Taylor flow in which local supersonic zones join on the axis of the nozzle in such a way that the sonic surface is orthogonal to this axis. The corresponding flow pattern is not unique in the three-dimensional motions of gas. We shall investigate, therefore, the possibility of another type of limiting Taylor flow where the transition surface from subsonic to supersonic velocities is tangent to the x-axis at the center of the nozzle. In this case it is necessary that

$$\partial u / \partial x = 0 \quad \text{when} \quad x = y = z = 0 \tag{1.5}$$

We shall consider the consequences derived from the condition (1.5). Assuming the stream to be analytical, we shall expand, in accordance with the representation above, the expression for the velocity potential Φ in the form of a power series

$$\Phi = \sum_{l,\,m,\,n}^{\infty} a_{l,\,2m,\,2n}\, x^l y^{2m} z^{2n} \tag{1.6}$$

where the coefficients a_{000}, a_{100} and a_{200} are given by the relationships

$$a_{000} = 0, \qquad a_{100} = 1, \qquad a_{200} = 0 \tag{1.7}$$

Substituting the series (1.6) into the equation of motion (1.4) and equating in the expression thus obtained the terms with the same powers x, y and z, it is possible to establish relationships which connect the coefficients $a_{l,2m,2n}$.

The equation obtained through correlation of the coefficients of $x^{\lambda}y^{2\mu}z^{2\nu}$ has the form

$$\begin{aligned}
&(\varkappa+1)\left[(\lambda+2)(\lambda+1)a_{\lambda+2,2\mu,2\nu}+(2\mu+2)(2\mu+1)a_{\lambda,2\mu+2,2\nu}+\right.\\
&\qquad\left.+(2\nu+2)(2\nu+1)a_{\lambda,2\mu,2\nu+2}\right]= \\
&=\sum_{(i+l+r=\lambda+4,\ j+m+s=\mu,\ k+n+t=\nu)} a_{i,2j,2k}a_{l,2m,2n}a_{r,2s,2t}\,ilr\,[i+l-2+(r-1)(\varkappa-1)]+\\
&+\sum_{(i+l+r=\lambda,\ j+m+s=\mu+2,\ k+n+t=\nu)} 8a_{i,2j,2k}a_{l,2m,2n}a_{r,2s,2t}\,jms\,[2(j+m-1)+(2s-1)(\varkappa-1)]+\\
&+\sum_{(i+l+r=\lambda,\ j+m+s=\mu,\ k+n+t=\nu+2)} 8a_{i,2j,2k}a_{l,2m,2n}a_{r,2s,2t}\,knt\,[2(k+n-1)+(2t-1)(\varkappa-1)]+\\
&+\sum_{(i+l+r=\lambda+2,\ j+m+s=\mu+1,\ k+n+t=\nu)} 2a_{i,2j,2k}a_{l,2m,2n}a_{r,2s,2t}\,\{2jmr\,[i+l+(r-1)(\varkappa-1)]+\\
&\qquad+ils\,[2(j+m)+(2s-1)(\varkappa-1)]\}+\\
&+\sum_{(i+l+r=\lambda+2,\ j+m+s=\mu,\ k+n+t=\nu+1)} 2a_{i,2j,2k}a_{l,2m,2n}a_{r,2s,2t}\,\{2knr\,[i+l+(r-1)(\varkappa-1)]+\\
&\qquad+ilt\,[2(k+n)+(2t-1)(\varkappa-1)]\}+\\
&+\sum_{(i+l+r=\lambda,\ j+m+s=\mu+1,\ k+n+t=\nu+1)} 8a_{i,2j,2k}a_{l,2m,2n}a_{r,2s,2t}\,\{kns\,[2(j+m)+(2s-1)(\varkappa-1)]+\\
&\qquad+jmt\,[2(k+n)+(2t-1)(\varkappa-1)]\}
\end{aligned} \tag{1.8}$$

2. Equation (1.8) which correlates the coefficients $a_{l,2m,2n}$ of the solution (1.6) is the unique description of the gas flow in the region of its center (the shape of the nozzle is not given *a priori*). Before we begin their detailed study, we shall investigate certain basic qualitative peculiarities of three-dimensional motions. For this we shall limit the number of terms in the expansion (1.6) to the fourth order inclusive. From the first five terms of Equation (1.8) we have

$$a_{020} + a_{002} = 0, \qquad a_{120} + a_{102} = 0, \qquad a_{220} + a_{202} = 0 \tag{2.1}$$

$$6a_{040} + a_{022} = 4a_{020}(a_{120} + a_{020}^2), \qquad 6a_{004} + a_{022} = 4a_{002}(a_{102} + a_{002}^2) \tag{2.2}$$

In a sufficiently small region of the plane $x = 0$, containing the origin of the coordinates, coefficients a_{020} and a_{002} characterize the magnitudes of the cross-sectional components of the stream velocity. In the same region the coefficients a_{120}, a_{102}, a_{220} and a_{202} give the values of the first and the second derivatives of the components v and w

along the x-axis, and determine the direction of the gas velocity near the plane $x = 0$.

From Equations (2.1) there follows that the quantities a_{020} and a_{002}, a_{120} and a_{102}, and a_{220} and a_{202} are of opposite sign.

Thus, in the plane $x = 0$ and near it, one of the cross-sectional components of the particle velocity is directed to the center of the flow, the other away from it, i.e. in one of the two planes of symmetry of gas motion the stream converges, in the other, it diverges. This picture is a distinguishing feature of three-dimensional flows, as in the plane and axisymmetrical motions a similar structure of the velocity field is impossible also because of the condition (1.5), as it is shown in [2] that all the coefficients $a_{r,2m,2n}$ ($r = 0, 1, 2$; $m, n = 0, 1, 2, \ldots$), except $a_{100} = 1$, become zero. Moreover, in this latter case the sonic surface becomes plane, namely, perpendicular to the axis of the channel, and within that surface any cross-sectional component of the stream velocity vanishes. In the general case of three-dimensional flows, solely as a consequence of condition (1.5), an infinite system of equations (1.8) is established, the first of which are Equations (2.1) and (2.2).

If we require, however, that in the first quadrant of the plane $x = 0$ the quantities v and w, $\partial v/\partial x$, $\partial w/\partial x$, $\partial^2 v/\partial x^2$ and $\partial^2 w/\partial x^2$ are to have the same signs, excluding a reversal of sign, i.e. that in this region of the plane in all directions either convergence or divergence takes place, then from Equations (2.1) and (2.2) we obtain immediately

$$a_{020} = a_{002} = a_{120} = a_{102} = a_{220} = a_{202} = a_{040} = a_{004} = a_{022} = 0 \qquad (2.3)$$

If one develops the expansion (1.6) for the potential Φ to the next higher terms (up to sixth order included), it may be shown that in this case also

$$a_{140} = a_{104} = a_{122} = a_{240} = a_{204} = a_{222} = 0 \qquad (2.4)$$

Thus, in the approximation considered in this paragraph the sonic surface is, as in the case of plane and axisymmetrical flows, a plane perpendicular to the axis of the channel. Both cross-sectional components of the gas-particle velocity vanish in this plane and, moreover, $\partial u/\partial x = 0$. Note that the assumptions about the signs of functions v and w, $\partial v/\partial x$ and $\partial w/\partial x$, $\partial^2 v/\partial x^2$ and $\partial^2 w/\partial x^2$ made above do not constitute a significant limitation, since in real nozzles convergence of the stream takes place in all directions up to the critical cross-section and after this, divergence.

3. We shall prove now the theorem pertaining to three-dimensional supersonic flows which serves as a strict proof of the results obtained in the preceding paragraph.

Theorem. Let a flow of an ideal gas be given, which has two mutually perpendicular planes of symmetry $y = 0$ and $z = 0$ and which is analytical at the point $x = y = z = 0$ and in some region K about this point. Further, let the stream velocity $u(x, 0, 0)$ along the x-axis approach the local velocity of sound at the point $x = 0$, also let the derivative of the velocity there vanish:

$$u = 1, \qquad \partial u / \partial x = 0 \qquad \text{for } x = y = z = 0 \tag{3.1}$$

Finally, let the quantities v and w, $\partial v/\partial x$ and $\partial w/\partial x$ $\partial^2 v/\partial x^2$ and $\partial^2 w/\partial x^2$ have the same signs in the first quadrant of the plane $x = 0$ without reversal of sign.

For all y and z in K we then have

$$u(0, y, z) = 1, \qquad \partial u(0, y, z) / \partial x = 0, \qquad v(0, y, z) = w(0, y, z) = 0 \tag{3.2}$$

From the assumptions of the theorem, Equations (1.7) are derived. Together with (2.3), we have

$$a_{000} = a_{020} = a_{002} = 0, \qquad a_{100} = 1, \quad a_{120} = a_{102} = 0; \qquad a_{200} = a_{220} = a_{202} = 0 \tag{3.3}$$

Equations (3.2) lead to the system

$$a_{0,2m,2n} = a_{1,2m,2n} = a_{2,2m,2n} = 0 \qquad (m, n = 0, 1, 2, \ldots) \tag{3.4}$$

for all x, y, z within K, with the exception of $a_{100} = 1$. The proof of the relationships (3.4) will be carried out by the method of exact mathematical induction. For this purpose we shall make the following assumptions:

$$a_{0,2m,2n} = a_{1,2m,2n} = a_{2,2m,2n} = 0 \qquad \text{for } m + n \leqslant \sigma - 1 \tag{3.5}$$

(with the exception of $a_{100} = 1$), and we shall show that Formulas (3.5) are valid also for $m + n = \sigma$.

In Equation (1.8) we shall assume $\lambda = \tau$ $(\tau = 0, 1, 2)$, $\mu + \nu = \sigma - 1$; using system (3.5), we obtain

$$(2\mu + 2)(2\mu + 1)\, a_{\tau,2(\mu+1),2\nu} + (2\nu + 2)(2\nu + 1)\, a_{\tau,2\mu,2(\nu+1)} = 0 \; (\mu + \nu = \sigma - 1, \tau = 0, 1, 2) \tag{3.6}$$

We shall write the expression which contains terms of the order of $2\sigma - 1$, and which are included in the expansions of functions $v(0, y, z)$, $w(0, y, z)$, $\partial v(0, y, z)/\partial x$, $\partial w(0, y, z)/\partial x$, $\partial^2 v(0, y, z)/\partial x^2$ and $\partial^2 w(0, y, z)/\partial x^2$, in the form of power series

$$\frac{1}{2(\tau!)} \frac{\partial^\tau v_{2\sigma-1}}{\partial x^\tau} = \sigma a_{\tau,2\sigma,0} y^{2\sigma-1} + (\sigma - 1)\, a_{\tau,2(\sigma-1),2} y^{2\sigma-3} z^2 + \ldots + a_{\tau,2,2(\sigma-1)}\, y z^{2(\sigma-1)} \tag{3.7}$$

$$\frac{1}{2(\tau!)} \frac{\partial^\tau w_{2\sigma-1}}{\partial x^\tau} = \sigma a_{\tau,0,2\sigma} z^{2\sigma-1} + (\sigma - 1)\, a_{\tau,2,2(\sigma-1)} z^{2\sigma-3} y^2 + \ldots + a_{\tau,2(\sigma-1),2}\, z y^{2(\sigma-1)}$$

In accordance with the requirement (3.5), Formulas (3.7) give the first terms of the expansions of the corresponding functions. Because in the first quadrant of the plane $x = 0$ the components of the stream velocity v and w and their first two derivatives with respect to x must, according to the assumption, have the same signs without reversal, the constants

$$a_{\tau,2\sigma,0} \quad a_{\tau,2,2(\sigma-1)}, \quad a_{\tau,0,2\sigma}, \quad a_{\tau,2(\sigma-1),2} \qquad (\tau = 0, 1, 2)$$

also must be of the same signs. But this condition contradicts the relationships (3.6). Hence, it follows that it is necessary that

$$a_{\tau,2\mu,2\nu} = 0 \qquad (\mu + \nu \leqslant \sigma, \ \tau = 0, 1, 2) \tag{3.8}$$

except $a_{100} = 1$. Since for $\sigma = 1$ and $\sigma = 2$ Equations (3.5) are identical with Equations (3.3), the theorem is fully proven. Note that in the case of plane flows and of flows with axial symmetry the conditions imposed on the signs of functions $v(0, y, z)$ and $w(0, y, z)$ and on their first derivatives with respect to x are fulfilled automatically. Thus, the theorem proven in this paragraph is a generalization for three-dimensional gas flows of the results of Frankl and Görtler [1,2].

4. As has been shown, the condition (3.1) in the general case of three-dimensional flows of a gas does not lead to the case that the surface of transition passing through the center of the flow becomes plane. Only if additional requirements are imposed, i.e. if a stream is subjected to either convergence or divergence from all sides in the region of the plane $x = 0$, are Formulas (3.2) the consequence of condition (3.1). In this case the velocity is equal to the sound velocity throughout the plane $x = 0$ while the cross-sectional velocity components vanish. Moreover, for $x = 0$ the derivatives $\partial v/\partial x$, $\partial w/\partial x$, $\partial^2 v/\partial x^2$ and $\partial^2 w/\partial x^2$ also become zero. But since the walls of the channel are formed by the stream lines, analytical mixed flows with a plane transition surface from subsonic to supersonic velocities are possible only in special nozzles with sufficiently gradual variation of the form of the walls in the region of the critical cross-section. Therefore, for the merging of the local supersonic zones at the nozzle axis, which are adjacent to its walls, the derivative $\partial u/\partial x$, generally speaking, is not equal to zero in the center of the flow, except when the stream is subjected to convergence in all directions up to the critical cross-section and thereafter to divergence. Corresponding solutions of the equations of gas motion have been studied earlier, where it was shown that in this case the region of supersonic velocities is bounded by the two surfaces which are orthogonal to the nozzle axis at the point of intersection with it and mutually tangent at this point [12].

In conclusion we shall note that only analytical flows have been considered here. The derived relations are not valid for flows with discontinuities in the derivatives of the velocity components.

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